ISSN 2229-5518

Representation, Normalization and Dimensionality of a Particle Wave Function for an Arbitrary One-Dimensional Potential

A. Zh. Khachatrian, Al. G. Aleksanyan, N. A. Aleksanyan

Abstract— The problem of determining of the normalization constant of a wave function describing an arbitrary type of an infinite motion for a one-dimensional potential is discussed. It is shown that for the cases of the momentum or the quasi-wave number representations the normalization constant does not depend on the representation parameter. In contrast to these cases for the energy representation the normalization constant depends on the energy state value. It is proved, that regardless of the representation choice and the form of a one-dimensional potential the normalization constant of a wave function for an arbitrary infinite motion coincides with the value of the normalization constant of the free motion. The connection between the normalization condition of the wave function and the magnitudes of the amplitudes determining the asymptotic behavior of the wave function is also established.

Index Terms—. one-dimensional motion, normalization problem, wave functions representations.

It is well known that one of the basic concepts of quantum mechanics is the normalization condition for the wave function, which is important for describing the probabilistic nature of displaying of certain values for the physical quantities characterizing the various properties of microscopic systems [1]. Thus, depending on a motion character, i.e. it takes place in a finite or an infinite area of space, the wave function must be normalized to unity or δ - function, respectively. The question is how the normalization constant depends on the values of the constants determining the motion type or the asymptotic behavior of a wave function. In this paper the detailed analysis of the normalization problem for an infinite one-dimensional motion in an arbitrary potential field is performed. The limitation for the reviewing given below is the assumption that in infinites the potential energy asymptotically tends to zero.

Often, when the problem of description of a one-dimensional quantum-mechanical motion is discussed, instead of an energy-depending wave function $\varphi_E(x)$ in order to specify the form of the solution (the problem statement), the wave function as a parametric function of the momentum $\varphi_p(x)$ or quasi-wave number $\varphi_k(x)$ is considered. In the general case a direction of a one-dimensional motion (the vector of a particle momentum) remains uncertain, but modules of the momentum p and quasi-wave number k are the well-defined quantities.

Regardless of the type of the asymptotic conditions depositing on a wave function all three representations of the wave function $\varphi_E(x)$, $\varphi_p(x)$ and $\varphi_k(x)$ satisfy to the same wave equation;

$$\frac{d^2\varphi_{E,p,k}(x)}{dx^2}+\frac{2m}{\hbar^2}\left(E-U(x)\right)\varphi_{E,p,k}(x)=0, \quad (1)$$

where

$$E=\frac{p^2}{2m}=\frac{\hbar^2k^2}{2m} \quad (2)$$

however, the dimensionalities of these three functions are differ from each other. As it follows from Eq. (1), the wave equation determines only the coordinate dependence but it leaves the dimensionality of a wave function uncertain. The dimensionality of a wave function should be defined from another condition differ from the condition according to which a wave function should satisfy to Eq. (1). Indeed, in accordance with the normalization condition of continuous spectrum states for the above-mentioned types of wave functions can be written:

$$\int_{-\infty}^{\infty}\varphi_E(x)\varphi_{E'}^*(x)dx=\delta(E-E') \quad (3)$$

$$\int_{-\infty}^{\infty}\varphi_p(x)\varphi_{p'}^*(x)dx=\delta(p-p') \quad (4)$$

$$\int_{-\infty}^{\infty}\varphi_k(x)\varphi_{k'}^*(x)dx=\delta(k-k') \quad (5)$$

Note that the dimensionality (further, we will mention dimensionality by the quadratic brackets) of the δ-function is the inverse of the dimensionality of its argument ($[\delta(x)]=1/[x]$), so for the dimensionalities of the wave functions $\varphi_E(x)$, $\varphi_p(x)$ and $\varphi_k(x)$ one can write:

$$[\varphi_E(x)]=1/[\sqrt{E}][\sqrt{x}] \quad (6)$$

$$[\varphi_p(E)]=1/[\sqrt{p}][\sqrt{x}] \quad (7)$$

$$[\varphi_k(E)]=1/[\sqrt{k}][\sqrt{x}] \quad (8)$$

As it follows from Eq. (8) since the quantity k has the dimensionality inverse of length $[k]=1/[x]$ the wave function $\varphi_k(x)$ is a dimensionless quantity;

$$[\varphi_k(x)]=[N] \quad (10)$$

where N is an arbitrary number.

For the most common form of an infinite motion the asymptotic behavior of a wave function can be written:

$$\varphi(x)=\begin{cases}a\,exp\{ikx\}+b\,exp\{-ikx\}, x\to-\infty,\\ c\,exp\{ikx\}+d\,exp\{-ikx\}, x\to+\infty.\end{cases} \quad (11)$$

- *A. Zh. Khachatrian,Head of Physics department of State Engineering University of Armenia, str. Teryan 105, Yerevan 0009, Armenia, E-mail: ashot. khachatrian@gmail.com*
- *Al. G. Aleksanyan, Senior researcher at Institute of Physics Applied Problems of NAS of Armenia, str. Nersessyan 25, Yerevan 0010, Armenia, E-mail: alalbert@inbox.ru*
- *N. A. Aleksanyan, PhD student at Physics department of State Engineering University Of Armenia, str. Teryan 105, Yerevan 0009, Armenia, E-mail: narek.aleksanyan@gmail.com*

For the magnitudes $k > 0$ the quantities a, d will be the amplitudes of waves converging to the potential and the quantities c, d will be the amplitudes of outgoing waves. A selection of any two amplitudes from the four above-mentioned amplitudes as originally given quantities completely determines the motion nature or the form of a wave function for the whole one-dimensional space. So, when in Eq. (11) $a = 1, d = 0$ are chosen then the wave function corresponds to the left-scattering problem, and when $a = 0,\ d = 1$ then the asymptotic behavior corresponds to the right-scattering problem. As it follows from Eq. (11) the dimensionalities of the amplitudes coincide with the dimensionality of a wave function of the corresponding representation. Using Eq. (6)-(8) one can write

$$[a_E(x)] = [d_E(x)] = [\varphi_E(x)], \tag{12}$$

$$[a_p(x)] = [d_p(x)] = [\varphi_p(x)], \tag{13}$$

$$[a_k(x)] = [d_k(x)] = [\varphi_k(x)]. \tag{14}$$

In accordance with Eq. (2) let us write down

$$E' = E(p') = p'^2/2m = E(k') = \hbar^2 k'^2/2m, \tag{15}$$

$$p' = p(E') = \sqrt{2mE'} = p(k') = \hbar k', \tag{16}$$

$$k' = k(p') = p'/\hbar = k(E') = \sqrt{2mE'}/\hbar. \tag{17}$$

Taking into account Eq. (15)-(17) and using equalities between the following three dimensionless quantities

$$\delta(E - E')dE = \delta(p - p')dp = \delta(k - k')dk \tag{18}$$

one can write

$$\delta(E - E') = \delta(p - p')\, dp/dE = \delta(k - k')\, dk/dE. \tag{19}$$

By using Eq. (19) and Eq. (3)-(5) it can be shown that between the corresponding representations of the wave functions normalized on a δ-function the following relations take place:

$$\varphi_E(x) = \sqrt{\frac{dk}{dE}}\varphi_k(x),\ \varphi_E(x) = \sqrt{\frac{dp}{dE}}\varphi_p(x),\ \varphi_p(x) = \sqrt{\frac{dk}{dp}}\varphi_k(x). \tag{20}$$

It is well known that for the case of the free motion in the quasi-wave number representation the wave function $\varphi_k(x)$ having $\delta(k - k')$ normalization (see, for example, [1]) has the form of:

$$\varphi_k(x) = \left(\frac{1}{2\pi}\right)^{1/2} \exp\{ikx\}, \tag{21}$$

From Eq. (20) for the functions $\varphi_p(x)$ and $\varphi_E(x)$ we will get

$$\varphi_p(x) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{1}{2}} \exp\{ikx\}, \tag{22}$$

$$\varphi_E(x) = \sqrt[4]{\frac{m}{2E}}\left(\frac{1}{2\pi\hbar}\right)^{\frac{1}{2}} \exp\{ikx\}. \tag{23}$$

As one can see from Eq. (17) - (19) the normalization constants for the functions $\varphi_k(x)$ and $\varphi_p(x)$ do not depend on the parameters of representation and are constants: $1/\sqrt{2\pi}$ and $1/\sqrt{2\pi\hbar}$, respectively. In contrast to them the normalization constant of the function $\varphi_E(x)$ depends on the parameter representation, and is in power $\sim(E)^{-1/4}$.

It is important to note that the mentioned values of normalization constants appear not only in the case of the free motion states. The relations between the normalization constants of the wave functions writing for difference representations have a universal form, which does not depend on a type of infinite motion (free motion, left or right scattering problems and so on) and on a shape of a one-dimensional potential. In accordance with the main result of the paper [2] obtained on the base of so called method of converging waves (see [3-5]) the magnitude of the normalization constant for the quasi-wave number representation does not depend on the form of a scattering potential and it is determined by the boundary conditions imposed on the character of the investigated motion. So, if for the amplitudes of the converging waves a, d the following equality takes place;

$$a_k a_k^* + d_k d_k^* = 1/2\pi, \tag{24}$$

then the wave function $\varphi_k(x)$ will be normalized on a $\delta(k - k')$ function (see Eq. (5)). As it follows from Eq. (20) and Eq. (21), for the wave functions $\varphi_E(x)$, $\varphi_p(x)$ the normalization conditions Eq. (3), Eq. (4) will be provided if for amplitudes of converging waves rewriting in the corresponding representations the following equalities take place;

$$a_E a_E^* + d_E d_E^* = \frac{1}{2\pi\hbar}\sqrt{\frac{m}{2E}}, \tag{25}$$

$$a_p a_p^* + d_p d_p^* = \frac{1}{2\pi\hbar}. \tag{26}$$

It is easy to see that from Eq. (24)-(26) the following relation between the amplitudes of converging waves of different representations can be obtained:

$$a_E = \sqrt[4]{\frac{m}{2E}}a_p = \left(\frac{1}{\hbar}\right)^{1/2}\sqrt[4]{\frac{m}{2E}}a_k,\ d_E = \sqrt[4]{\frac{m}{2E}}d_p = \left(\frac{1}{\hbar}\right)^{1/2}\sqrt[4]{\frac{m}{2E}}d_k. \tag{27}$$

This result has an absolutely general character and it is true for any asymptotic behavior of a wave function. So, for example, as it follows from Eq. (21)-(23) for the wave function of the free motion when a particle moves in positive direction the corresponding amplitudes have the form of:

$$a_k = \left(\frac{1}{2\pi}\right)^{1/2},\ a_p = \left(\frac{1}{2\pi\hbar}\right)^{1/2},\ a_E = \left(\frac{1}{2\pi\hbar}\right)^{1/2}\sqrt[4]{\frac{m}{2E}} \tag{28}$$

and $d_E = d_p = d_k = 0$

From Eq. (27)-(28) one can conclude that regardless of the representation choice and the potential form the normalization constant of a wave function for an arbitrary infinite motion coincides with the value of the normalization constant of the free motion.

So, we have shown that for an arbitrary one dimensional motion an explicit connection between the boundary condition and the normalization condition of a wave function can be established. We deduced the corresponding formulas (see Eq. (24)-(26)) for three types of a wave function representation.

At first glance it may seem that the presented results are of a purely methodological interest. However, apart from the methodological interest these results are also of great practical importance. So, for description of evolutionary behavior of wave packets propagating thought a one –dimensional media it is very important to have a convenient basis of expansions consisted of normalized functions [6-10]. Particularly, we found the normalization constant of the wave functions describing the scattering process for an arbitrary one dimensional field.

REFERENCES

[1] David Bohm. *Quantum Theory*. Prentice-Hall, New York, 1952

[2] A.Zh. Khachatrian, V.A. Khoectyan, N.A. Aleksanyan. Prog. of Conf.: Physics of Low Dimensional Semiconductor Systems, p. 139-151, Yerevan 2012, Armenia

[3] D.M. Sedrakian, *A.Zh. Khachatrian*, *Physics Letters* A, 265, 294 (2000)

[4] A. Zh. Khachatrian. Armenian Journal of Physics, 3, 178 (2010)

[5] A. Zh. Khachatrian. Armenian Journal of Physics, 4, 90 (2011)

[6] J. S. Bell. Speakable and unspeakable in quantum mechanics. Cambridge (2004)

[7] P. Bokes. Phys. Rev. A 83, 032104 (2011)

[8] T. E. Hartman. J. Appl. Phys. 33,3427 (1962)

[9] E. H. Hauge, J. A. Stuvneng. Rev. Mod. Phys. 61, 917 (1989)

[10] R. Landauer, Th. Martin. Rev. Mod. Phys. 66, 217 (1994)